# Near Miss $abc$-Triples in General Number Fields

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**Abstract**

Masser and others have constructed sequences of "near miss" $abc$-triples, i.e., triples of relatively prime rational integers $(a, b, c)$ that asymptotically come close to violating the inequality that appears in the $abc$ Conjecture. In the present paper, we show various partial generalizations of Masser's result to arbitrary number fields.

## Contents



# 1 Introduction

Masser proved the following theorem ([3]). We refer to Notation 1.2 and Definition 1.3 concerning the notation and terminology that appears.

**Theorem 1.1 (Masser).** *Let $P_0$, $\gamma \in \mathbb{R}_{>0}$ be (positive real numbers) such that $\gamma < \dfrac{1}{2}$. Then there exists a strict abc-triple $(a, b, c)$ in (the field of rational numbers) $\mathbb{Q}$ whose conductor $P_{\mathbb{Q}}(a, b, c)$ satisfies the following conditions:*

- $P_{\mathbb{Q}}(a, b, c) > P_0$;
- $|abc| > P_{\mathbb{Q}}(a, b, c)^3 \exp\left((\log P_{\mathbb{Q}}(a, b, c))^{\frac{1}{2}-\gamma}\right)$.

In the present paper, we show the existence of an $abc$-triple in an arbitrary number field $L$ that satisfies similar (but slightly weaker) inequalities to the inequalities in Theorem 1.1. The inequalities that we obtain are weaker than the inequalities of Theorem 1.1 in the following two respects: the quantity on the left-hand side of this second inequality will be replaced by the "height" of the triple, while the quantity on the right-hand side of this second inequality will be replaced by a quantity of slightly lower order. Moreover, we show, in the case of a quite substantial class of number fields "$L$", that the *abc*-triple whose existence is asserted may be chosen to satisfy the condition that it does not arise (even after possible multiplication by a scalar) from an *abc*-triple that is contained in some proper subfield of $L$.

The strategy applied in Masser's proof of Theorem 1.1 is to construct an *abc*-triple such that the prime numbers dividing $a$ or $b$ are bounded, while $c$ is divisible by a large power of a fixed prime number; these conditions on the *abc*-triple imply that $P_{\mathbb{Q}}(a,b,c)$ is relatively small. In the present paper, we give generalizations of this argument of Masser in two cases, each of which applies to number fields $L$ more general than $\mathbb{Q}$. One is the case where the rank (as a finitely generated abelian group) of the group of units $\mathcal{O}_L^{\times}$ of $L$ is 0, i.e., the case where $L$ is either the field of rational numbers or an imaginary quadratic field. In this case, a suitable analogue of the triangle inequality holds. Such an analogue of the triangle inequality allows us to mimick Masser's proof and hence to obtain bounds on the "size" of the *abc*-triple in terms of $P_{\mathbb{Q}}(a,b,c)$ (cf. Theorem A). The other case is the case where the rank of the group of units $\mathcal{O}_L^{\times}$ is positive. In this case, by considering suitable powers of a given non-torsion element of $\mathcal{O}_L^{\times}$, we construct *abc*-triples that satisfy the desired inequalities (cf. Theorem B).

**Notation 1.2.**

(1) For a finite set $X$, we shall use the notation $\#X$ to denote the cardinality of $X$.

(2) For an algebraic number field $L$, we use the notation $\mathcal{O}_L$ (resp. $L^{\times}$, $\mathcal{O}_L^{\times}$, $\mu_L$, $\mathrm{rk}_L$, $h_L$) to denote the ring of integers of $L$ (resp. the multiplicative group of $L$, the group of units of $L$, the group of roots of unity of $L$, the rank of the finitely generated abelian group $\mathcal{O}_L^{\times}$, the class number of $L$).

(3) $\mathbb{V}(L)$ (resp. $\mathbb{V}^{\mathrm{arc}}(L)$, $\mathbb{V}^{\mathrm{non}}(L)$) denotes the set of places (resp. archimedean places, non-archimedean places) on L. For $v \in \mathbb{V}^{\mathrm{non}}(L)$, $\mathfrak{p}_v$ denotes the maximal ideal of $\mathcal{O}_L$ associated to $v$, and $p_v$ denotes the residue characteristic of $v$.

(4) $\mathrm{N}_L$ denotes the absolute norm on L, i.e., for an ideal $\mathfrak{a} \subset \mathcal{O}_L, \mathrm{N}_L(\mathfrak{a}) = \#(\mathcal{O}_L/\mathfrak{a})$, and for an element $a \in \mathcal{O}_L$, $\mathrm{N}_L(a) = \mathrm{N}_L(a\mathcal{O}_L)$.

(5) For $x$ an element of a topological field isomorphic to $\mathbb{R}$ or $\mathbb{C}$, $|x|$ denotes the usual absolute value, i.e., if $x \neq 0$, then $x/|x|$ is a unit with respect to the topology. If $v \in \mathbb{V}(L)^{\mathrm{arc}}$, then, for $x \in L^{\times}$, $||x||_v := |x|^{[L_v:\mathbb{R}]}$, where $L_v$ denotes the completion of $L$ with respect to $v$ (so $L_v \cong \mathbb{R}$ or $L_v \cong \mathbb{C}$), and $x$ is considered as an element of $L_v$. If $v \in \mathbb{V}^{\mathrm{non}}(L)$, then, for $x \in L^{\times}$, $||x||_v := \mathrm{N}_L(\mathfrak{p}_v)^{-\mathrm{ord}_v(x)}$, where $\mathrm{ord}_v(x) \in \mathbb{Z}$ denotes the unique element $\in \mathbb{Z}$ such that the fractional ideal $x \cdot \mathfrak{p}_v^{-\mathrm{ord}_v(x)}$ is generated by $v$-adic units $\in L$.

**Definition 1.3.**

(1) Let $a, b, c \in L\setminus\{0\}$. If $a+b+c=0$, then we say that $(a,b,c)$ is an *abc-triple*. For $a, b, c \in L$, if $a, b, c \in \mathcal{O}_L$ and $a\mathcal{O}_L + b\mathcal{O}_L + c\mathcal{O}_L = \mathcal{O}_L$, then we say that $a, b, c$ are *relatively prime*. For an *abc*-triple $(a, b, c)$, if $a$, $b$, $c$ are relatively prime, then we shall say that $(a, b, c)$ is a *strict abc-triple*. Note that some authors use the term "*abc*-triple" to refer to a "strict *abc*-triple", as defined in the present paper.

(2) For an *abc*-triple $(a, b, c)$, we define the conductor

$$P_L(a,b,c) := \prod_{\substack{v\in\mathbb{V}^{\mathrm{non}}(L)\\ \#\{||a||_v,||b||_v,||c||_v\}\geq 2}} \mathrm{N}_L(\mathfrak{p}_v).$$

Note that if $(a, b, c)$ is a strict $(a, b, c)$-triple, then

$$P_L(a,b,c) = \prod_{\substack{v\in\mathbb{V}^{\mathrm{non}}(L)\\ ||abc||_v<1}} \mathrm{N}_L(\mathfrak{p}_v).$$

(3) For an *abc*-triple $(a, b, c)$, we define

$$H_L(a,b,c) := \prod_{v\in\mathbb{V}(L)} \max\{||a||_v,||b||_v,||c||_v\}.$$

and call it the *height* of $(a, b, c)$ (cf. [4, §2]). Note that if $(a, b, c)$ is a strict $(a, b, c)$-triple, then

$$H_L(a,b,c) = \prod_{v\in\mathbb{V}^{\mathrm{arc}}(L)} \max\{||a||_v,||b||_v,||c||_v\}.$$

The main theorems of the present paper are the following.

**Theorem A.** *Let $L$ be an imaginary quadratic field (which we regard as a subfield of $L_v \cong \mathbb{C}$, where $v$ denotes the unique element of $\mathbb{V}^{\mathrm{arc}}(L)$) and $P_0, \gamma \in \mathbb{R}_{>0}$ be such that $\gamma < \frac{1}{2}$. Then there exists a strict abc-triple $(a, b, c)$ in $L$ such that*

- $P_L(a,b,c) > P_0$;
- $|abc|^2 > P_L(a,b,c)^3 \exp\left((\log P_L(a,b,c))^{\frac{1}{2}-\gamma}\right)$.

**Theorem B.** *Let $L$ be a number field, $u_0 \in \mathcal{O}_L^\times \setminus \mu_L$, and $P_0$, $\delta \in \mathbb{R}_{>0}$ such that $\delta < 1$. Then there exists a positive integer $l$ such that if we set $u := u_0^l$, $a := -1$, $b := u$, $c := 1-u$, then the following conditions are satisfied:*

- *$(a, b, c)$ is a strict abc-triple;*
- $P_L(a,b,c) > P_0$;

- $H_L(a,b,c) > P_L(a,b,c)(\log P_L(a,b,c))^{1-\delta}$.

In fact, Theorem A would be somewhat more meaningful if the $(a,b,c)$ in the statement of Theorem A could be chosen in such a way that the following condition on $(a,b,c)$ is satsfied:

$(*_{\mathbb{Q}})$ $(a,b,c)$ does not arise (even after possible multiplication by a scalar) from an *abc*-triple that is contained in $\mathbb{Q}$, i.e., $\frac{b}{a}$ is not contained in $\mathbb{Q}$.

Indeed, it is easy to verify (cf. the argument given below for more details) that Theorem A in its present form (i.e., in which the condition $(*_{\mathbb{Q}})$ is *not necessarily* satisfied) follows immediately from Masser's result (i.e., Theorem 1.1), which yields *abc*-triples that do *not* satisfy $(*_{\mathbb{Q}})$. In a similar vein, we observe that, in Theorem B, it is of interest to know whether or not $u$ can be chosen so that $u$ is not contained in any proper subfield of $L$.

With regard to Theorem A, we remark that the argument given in the present paper is *insufficient* from the point of view of guaranteeing that $(a,b,c)$ may be chosen so that $(*_{\mathbb{Q}})$ is satisfied. Nevertheless, we included Theorem A in the present paper in the hope that some relatively minor modification of the argument given in the present paper may be sufficient to prove a variant of Theorem A of the desired form (i.e., that asserts that $(a,b,c)$ may be chosen so that $(*_{\mathbb{Q}})$ is satisfied).

Theorem A may be deduced from Masser's result as follows. (This explanation is of course different from the proof of Theorem A given in Section 3.) If $L$ is an imaginary quadratic field, and $(a,b,c)$ is a strict *abc*-triple as in Theorem 1.1, i.e.,

- $P_{\mathbb{Q}}(a,b,c) > P_0$;
- $|abc| > P_{\mathbb{Q}}(a,b,c)^3 \exp\left((\log P_{\mathbb{Q}}(a,b,c))^{\frac{1}{2}-\gamma}\right)$,

then, since for $v \in \mathbb{V}^{\mathrm{non}}(L)$ such that $p_v | abc$, $\#\{||a||_v, ||b||_v, ||c||_v\} \geq 2$ and $\mathrm{N}_L(\mathfrak{p}_v) \geq p_v$, it follows that

$$P_L(a,b,c) \geq P_{\mathbb{Q}}(a,b,c) > P_0.$$

On the other hand, since for any prime number $p$,

$$p^2 \geq \prod_{\substack{v \in \mathbb{V}^{\mathrm{non}}(L) \\ \mathfrak{p}_v | p}} N(\mathfrak{p}_v),$$

it follows that

$$\begin{aligned} |abc|^2 &\geq P_{\mathbb{Q}}(a,b,c)^6 \exp\left(2(\log P_{\mathbb{Q}}(a,b,c))^{\frac{1}{2}-\gamma}\right) \\ &\geq P_{\mathbb{Q}}(a,b,c)^6 \exp\left((\log P_{\mathbb{Q}}(a,b,c)^2)^{\frac{1}{2}-\gamma}\right) \\ &\geq P_L(a,b,c)^3 \exp\left((\log P_L(a,b,c))^{\frac{1}{2}-\gamma}\right). \end{aligned}$$

With regard to Theorem B, we have the following Corollary C, which may be regarded as a refined version of Theorem B in the sense that it states that there exist strict *abc*-triples as in of Theorem B that do not arise by applying Theorem B to some subfield $L'$ of the given number field $L$ for which $\mathrm{rk}_{L'} < \mathrm{rk}_L$.

**Corollary C.** *Let $L$ be a number field which is neither the field of rational numbers nor an imaginary quadratic field, and $P_0, \delta \in \mathbb{R}_{>0}$ such that $\delta < 1$. Then there exists a unit $u \in \mathcal{O}_L^\times$ such that if we set $a := -1$, $b := u$, $c := 1 - u$, then the following conditions are satisfied:*

- $(a, b, c)$ *is an abc-triple;*
- $P_L(a, b, c) > P_0$*;*
- $H_L(a, b, c) > P_L(a, b, c)(\log P_L(a, b, c))^{1-\delta}$*;*
- *if* $L' \subset L$ *is a subfield such that* $\mathrm{rk}_{L'} < \mathrm{rk}_L$*, then* $u \notin L'$*.*

*In particular, if $L$ is unit-nondegenerate (see Definition 1.4 and Proposition D below), then $\frac{b}{a} = -u \notin L'$ for any proper subfield $L' \subsetneq L$.*

**Definition 1.4.** Let $L$ be a number field. If for any proper subfield $L' \subsetneq L$, $\mathrm{rk}_{L'} < \mathrm{rk}_L$, then we say that $L$ is *unit-nondegenerate*. Otherwise, we say that $L$ is *unit-degenerate*.

**Proposition D.** *Let $L$ be a totally imaginary Galois extension of $\mathbb{Q}$. Then the following hold:*

*(i) $L$ is unit-nondegenerate if and only if for each $v \in \mathbb{V}^{\mathrm{arc}}(L)$, the decomposition group of $v$ in $\mathrm{Gal}(L/\mathbb{Q})$ is not contained in the center of $\mathrm{Gal}(L/\mathbb{Q})$.*

*(ii) If $L$ is unit-nondegenerate and $M$ is a Galois extension of $\mathbb{Q}$ containing $L$, then $M$ is also unit-nondegenerate.*

**Example 1.5.** $L := \mathbb{Q}\left(\sqrt[3]{2}, \exp\left(\frac{2}{3}\pi i\right)\right) \subset \mathbb{C}$ is a Galois extension of $\mathbb{Q}$ with center-free Galois group (i.e., the symmetric group on 3 letters). Thus, by Proposition D, any Galois extension of $\mathbb{Q}$ containing $L$ is unit-nondegenerate.

In conclusion, for a quite substantial class of number fields $L$, we can find *abc*-triples that do not arise from any proper subfield of $L$, and that yield counterexamples of the "$\gamma = 0$ version" of the *abc* Conjecture for $L$.

## 2 Estimates for Ideal Counting Functions

Let $L$ be a number field.

**Definition 2.1.** Let $x, y \in \mathbb{R}_{>0}$ and $\mathfrak{a} \subset \mathcal{O}_L$ a non-zero ideal.

(1) If $\mathfrak{a} \neq \mathcal{O}_L$, then $\mathrm{LPN}(\mathfrak{a}) := \max\{\mathrm{N}_L(\mathfrak{p}_v) \mid v \in \mathbb{V}^{\mathrm{non}}(L),\ \mathfrak{p}_v | \mathfrak{a}\}$. We define $\mathrm{LPN}(\mathcal{O}_L) := 1$. For $a \in \mathcal{O}_L$, we define $\mathrm{LPN}(a) := \mathrm{LPN}(a\mathcal{O}_L)$.

(2) We define

$$\Pi_L(x) := \{\mathfrak{p} \subset \mathcal{O}_L : \text{non-zero prime ideal} \mid \mathrm{N}_L(\mathfrak{p}) \leq x\}$$

and

$$\pi_L(x) := \#\Pi_L(x).$$

(3) We define

$$\Psi_L(x, y) := \#\{\mathfrak{b} \subsetneq \mathcal{O}_L : \text{non-zero ideal} \mid \mathrm{N}_L(\mathfrak{b}) \leq x,\ \mathrm{LPN}(\mathfrak{b}) \leq y\}.$$

(4) We define

$$\Psi_L(x, y; \mathfrak{a}) := \#\{\mathfrak{b} \subsetneq \mathcal{O}_L : \text{non-zero ideal} \mid \mathrm{N}_L(\mathfrak{b}) \leq x,\ \mathrm{LPN}(\mathfrak{b}) \leq y,\ \mathfrak{a}+\mathfrak{b} = \mathcal{O}_L\}.$$

(5) We define

$$\theta_L(x) := \sum_{\substack{v \in \mathbb{V}^{\mathrm{non}}(L) \\ \mathrm{N}_L(\mathfrak{p}_v) \leq x}} \log \mathrm{N}_L(\mathfrak{p}).$$

The following lemma gives an estimate for $\theta$ (cf. [2, Satz 190], or, alternatively, [1, §3.2]; [5, Corollary 3.3]; [5, Corollary 3.4]). In the remainder of the present paper, we use the notation "$O(-)$" as it is defined in [5, Definition 1.4].

**Lemma 2.2.** *Let $x \in \mathbb{R}_{\geq 2}$. Then the following estimates hold:*

*(1) There exists a $C \in \mathbb{R}_{>0}$ such that*

$$\pi_L(x) = \int_2^x \frac{dt}{\log t} + O\left(x \exp\left(-C(\log x)^{\frac{1}{2}}\right)\right).$$

*(2)*

$$\pi_L(x) = \frac{x}{\log x} + \frac{x}{(\log x)^2} + O\left(\frac{x}{(\log x)^3}\right).$$

*(3)*

$$\theta_L(x) = x + O\left(\frac{x}{(\log x)^2}\right).$$

**Proof.** (1) See [1, §3.2].

(2) This estimate follows from (1) and the same elementary calculation as in the proof of [5, Corollary 3.3].

(3) This estimate follows from (2) and the same elementary calculation as in the proof of [5, Corollary 3.4].

□

We also have an estimate for the function $\Psi$.

**Lemma 2.3.** *Let $x, y, \gamma \in \mathbb{R}_{>0}, u \in \mathbb{Z}_{\geq 1}$, and $\mathfrak{q}_1, \ldots, \mathfrak{q}_u$ maximal ideals of $\mathcal{O}_L$ such that $\gamma < 1$, $x \geq 1$, and $\mathrm{N}_L(\mathfrak{q}_i) \leq y = (\log x)^\gamma$ for $i = 1, \ldots, u$. Write $D := \prod_{i=1}^{u} \mathfrak{q}_i$. Then the following estimate holds:*

$$\begin{aligned}\Psi_L(x, y; D) &= \exp\left(\left(\frac{1}{\gamma} - 1\right) y + \frac{y}{\gamma \log y} + O_{\gamma,u}\left(\frac{y}{(\log y)^2}\right)\right) \\ &= \exp\left(\left(\frac{1}{\gamma} - 1\right)(\log x)^\gamma + \frac{(\log x)^\gamma}{\gamma^2 \log\log x} + O_{\gamma,u}\left(\frac{(\log x)^\gamma}{(\log\log x)^2}\right)\right).\end{aligned}$$

**Proof.** Similar to the proof of [5, Theorem 3.9]. In the present situation, however, we observe that the statement of [5, Proposition 3.5] should be replaced by the following:

Let $x \in \mathbb{R}_{>1}$, $y \in \mathbb{R}_{\geq 2}$. Then the following inequality holds:

$$\begin{aligned}\frac{(\log x)^{\pi_L(y)}}{\pi_L(y)! \cdot \prod_{\mathfrak{p} \in \Pi_L(y)} \log \mathrm{N}_L(\mathfrak{p})} &\leq \Psi_L(x, y) + 1 \\ &\leq \frac{(\log x)^{\pi_L(y)}}{\pi_L(y)! \cdot \prod_{\mathfrak{p} \in \Pi_L(y)} \log \mathrm{N}_L(\mathfrak{p})} \left(1 + \sum_{\mathfrak{p} \in \Pi_L(y)} \frac{\log \mathrm{N}_L(p)}{\log x}\right)^{\pi_L(y)}.\end{aligned}$$

□

# 3 The Case of Imaginary Quadratic Fields

In this section, we prove Theorem A. Let $L$ be an imaginary quadratic field. Let $\delta \in \mathbb{R}_{>0}$, $\delta' \in \mathbb{R}_{>0}$ be such that

$$\delta < 12, \ \delta' < 12, \ \tfrac{12-\delta}{(h_L+\delta)^{\frac{1}{2}}} > 12 - \delta'.$$

Let $q$ be the smallest prime number such that $q^2 > P_0$ and $\mathfrak{q} := q\mathcal{O}_L$ is a maximal ideal. (Note that the existence of such a $q$ follows from Chebotarev's Density Theorem.) In the following argument, we shall make a suitable choice of

$$x_0 \in \mathbb{R}_{>3}$$

satisfying certain conditions that depend only on $L$, $P_0$ (e.g., via a dependence on $q$), $\delta$, and $\delta'$. Let $x$ be an element of $\mathbb{R}_{>x_0}$. We define $y = y(x) := (\log x)^{\frac{1}{2}} \geq 1$ and $G = G(x) := 1 + \lfloor \log x \rfloor > \log x \geq 1$. (Thus, $G \leq 1 + \log x \leq 2 \log x$.) Next, observe that it follows from Lemma 2.3 that by taking $x_0$ to be suitably large (in a way that depends only on $L$ and $P_0$), we may assume that $\Psi_L(x, y; \mathfrak{q})/G > q^2$. In particular, there exists a unique element $I = I(x) \geq 1$ of $\mathbb{Z}$ such that $0 < \frac{1}{q^2}\Psi_L(x, y; \mathfrak{q}) \leq Gq^{2I} < \Psi_L(x, y; \mathfrak{q})$.

**Lemma 3.1.** *For any $x \in \mathbb{R}_{>x_0}$, there exists a pair $(a_1, b_1)$ of elements of $\mathcal{O}_L$ such that*

*(1)* $\mathrm{LPN}(a_1) \leq y,\ \mathrm{LPN}(b_1) \leq y,$

*(2)* $a_1\mathcal{O}_L + b_1\mathcal{O}_L = \mathcal{O}_L,\ a_1\mathcal{O}_L + \mathfrak{q} = \mathcal{O}_L,\ b_1\mathcal{O}_L + \mathfrak{q} = \mathcal{O}_L,$

*(3)* $|a_1|^2 \leq x^{h_L},\ |b_1|^2 \leq x^{h_L},$

*(4)* $|a_1|^2 \leq |b_1|^2 \leq \exp(h_L)|a_1|^2,$

*(5)* $b_1 - a_1 \in \mathfrak{q}^I,$

*(6)* $a_1 \neq b_1.$

**Proof.** First, we observe that since $Gq^{2I} < \Psi_L(x, y; \mathfrak{q})$, there exist $Gq^{2I} + 1$ distinct ideals $\mathfrak{a}_0,\ \mathfrak{a}_1, \ldots,\ \mathfrak{a}_{Gq^{2I}} \subsetneq \mathcal{O}_L$ such that $\mathrm{N}_L(\mathfrak{a}_i) \leq x,\ \mathrm{LPN}(\mathfrak{a}_i) \leq y,$ and $\mathfrak{a}_i + \mathfrak{q} = \mathcal{O}_L$ for $i = 0,\ 1, \ldots,\ Gq^{2I}$. Since the $h_L$-th power of every ideal $\mathcal{O}_L$ is principal, there exists a generator $s'_0$ (resp. $s'_1, \ldots,\ s'_{Gq^{2I}}$) of $\mathfrak{a}_0^{h_L}$ (resp. $\mathfrak{a}_1^{h_L}, \ldots,\ \mathfrak{a}_{Gq^{2I}}^{h_L}$). Then since $\#(\mathcal{O}_L/\mathfrak{q}^I) = q^{2I}$, it follows from the Box Principle that there exist distinct elements $s_0,\ s_1, \ldots,\ s_G \in \{s'_0,\ s'_1, \ldots,\ s'_{Gq^{2I}}\} \subset \mathcal{O}_L$ such that

- $\mathrm{LPN}(s_i) \leq y$ for $i = 0, 1, \ldots, G,$
- $s_i\mathcal{O}_L + \mathfrak{q} = \mathcal{O}_L$ for $i = 0, 1, \ldots, G,$
- $s_i - s_j \in \mathfrak{q}^I$ for $i, j = 0, 1, \ldots, G.$

By reordering, we may suppose that

$$1 \leq \mathrm{N}_L(s_0) \leq \mathrm{N}_L(s_1) \leq \cdots \leq \mathrm{N}_L(s_G) \leq x^{h_L}.$$

If $x^{\frac{h_L}{\log x}}\mathrm{N}_L(s_i) < \mathrm{N}_L(s_{i+1})$ for $i = 0, 1, \ldots, G-1$, then since $G > \log x$, it follows that

$$x^{h_L} \leq x^{h_L}\mathrm{N}_L(s_0) < x^{\frac{Gh_L}{\log x}}\mathrm{N}_L(s_0) < \cdots < x^{\frac{h_L}{\log x}}\mathrm{N}_L(s_{G-1}) < \mathrm{N}_L(s_G) \leq x^{h_L},$$

a contradiction. Thus, there exists an $i_0 \in \mathbb{Z}$ such that $0 \leq i_0 \leq G-1$, and

$$\mathrm{N}_L(s_{i_0}) \leq \mathrm{N}_L(s_{i_0+1}) \leq x^{\frac{h_L}{\log x}}\mathrm{N}_L(s_{i_0}) = \exp(h_L)\mathrm{N}_L(s_{i_0}).$$

Since the ideals $s_{i_0}\mathcal{O}_L$ and $s_{i_0+1}\mathcal{O}_L$ are $h_L$-th powers, the ideal $s_{i_0}\mathcal{O}_L + s_{i_0+1}\mathcal{O}_L$ is also an $h_L$-th power, hence principal. Thus, there exists a pair $(a_1, b_1)$ of elements of $\mathcal{O}_L$ such that

$$a_1\mathcal{O}_L + b_1\mathcal{O}_L = \mathcal{O}_L,$$

$$\frac{b_1}{a_1} = \frac{s_{i_0+1}}{s_{i_0}}.$$

Then since $N_L(a_1) = |a_1|^2$ and $N_L(b_1) = |b_1|^2$, one verifies immediately that $(a_1, b_1)$ satisfies the conditions in the statement of Lemma 3.1.

$\square$

**Lemma 3.2.** *There exists a strict abc-triple* $(a, b, c)$ *such that*

*(1)*

$$P_L(a, b, c) > P_0,$$

*(2)*

$$H_L(a, b, c)^3 \geq \mathrm{N}_L(abc) = |abc|^2 > P_L(a, b, c)^3 \exp\left((12 - \delta')\frac{(\log P_L(a, b, c))^{\frac{1}{2}}}{\log\log P_L(a, b, c)}\right).$$

**Proof.** It follows from Lemma 3.1 that for any $x \in \mathbb{R}_{>x_0}$, there exists a pair $(a_1, b_1)$ of elements of $\mathcal{O}_L$ which satisfies the conditions in the statement of Lemma 3.1. Let

$$a := a_1, \ b := -b_1, \ c := -a_1 + b_1.$$

It follows from conditions (2) and (6) of Lemma 3.1 that $(a, b, c)$ is a strict *abc*-triple. Since $I \geq 1$, it follows from condition (5) of Lemma 3.1 that $c \in \mathfrak{q}$. Since, moreover, $\mathfrak{q} = q\mathcal{O}_L$ is a maximal ideal and $q^2 > P_0$, it follows that

$$P_L(a, b, c) \geq \mathrm{N}_L(\mathfrak{q}) = q^2 > P_0,$$

i.e., condition (1) of Lemma 3.2 is satisfied.

It remains to show that, for a suitable choice of $x_0$, $(a, b, c)$ satisfies condition (2) of Lemma 3.2. Since $(a, b, c)$ is a strict *abc*-triple, it follows from Definition 1.3 (3) that

$$\begin{aligned} H_L(a, b, c) &= \max\{\mathrm{N}_L(a), \ \mathrm{N}_L(b), \ \mathrm{N}_L(c)\} \\ &= \max\{|a|^2, \ |b|^2, \ |c|^2\}, \end{aligned}$$

and hence that $H_L(a, b, c)^3 \geq |abc|^2$. On the other hand, since, by conditions (1) and (5) of Lemma 3.1, $\mathrm{LPN}(a) \leq y$, $\mathrm{LPN}(b) \leq y$, and $c \in \mathfrak{q}^I$, it follows that

$$\begin{aligned} P_L(a, b, c) &= \left(\prod_{\substack{v \in \mathbb{V}^{\mathrm{non}}(L) \\ \mathfrak{p}_v | ab}} \mathrm{N}_L(\mathfrak{p}_v)\right)\left(\prod_{\substack{v \in \mathbb{V}^{\mathrm{non}}(L) \\ \mathfrak{p}_v | c}} \mathrm{N}_L(\mathfrak{p}_v)\right) \\ &\leq \exp(\theta_L(y)) \cdot \frac{\mathrm{N}_L(c)}{q^{2(I-1)}} \\ &= \exp(\theta_L(y)) \cdot \frac{|c|^2}{q^{2(I-1)}}. \end{aligned}$$

Thus, it follows from this estimate, together with condition (4) of Lemma 3.1 and the triangle inequality, that

$$|c|^2 \leq \left(1 + \exp\left(\tfrac{1}{2}h_L\right)\right)^2 |a|^2 \leq \left(1 + \exp\left(\tfrac{1}{2}h_L\right)\right)^2 |b|^2,$$

and hence that

$$\begin{aligned}|abc|^2 &\geq \frac{|c|^6}{\left(1+\exp\left(\frac{1}{2}h_L\right)\right)^4} \\ &\geq \frac{q^{6(I-1)}}{\left(1+\exp\left(\frac{1}{2}h_L\right)\right)^4}\exp(-\theta_L(y))^3 P_L(a,b,c)^3.\end{aligned}$$

Next, recall that it follows from the definition of $I$ and $G$ that

$$q^{2I} \geq \frac{\Psi_L(x,y;\mathfrak{q})}{Gq^2} \geq \frac{\Psi_L(x,y;\mathfrak{q})}{2q^2\log x}.$$

Therefore, if we write

$$C := \frac{1}{8q^{12}\left(1+\exp\left(\frac{1}{2}h_L\right)\right)^4},$$

then it follows that

$$|abc|^2 \geq C\left(\frac{\exp(-\theta_L(y))\Psi_L(x,y;\mathfrak{q})P_L(a,b,c)}{\log x}\right)^3.$$

Note that C depends only on $L$ and $P_0$. On the other hand, it follows from Lemma 2.2 (3) and Lemma 2.3 that

$$\begin{aligned}&\frac{\exp(-\theta_L(y))\Psi_L(x,y;\mathfrak{q})}{\log x} \\ &= \exp\left(-(\log x)^{\frac{1}{2}} + O\left(\frac{(\log x)^{\frac{1}{2}}}{(\log\log x)^2}\right)\right) \\ &\quad \cdot \exp\left((\log x)^{\frac{1}{2}} + \frac{4(\log x)^{\frac{1}{2}}}{\log\log x} + O\left(\frac{(\log x)^{\frac{1}{2}}}{(\log\log x)^2}\right)\right)\cdot\exp(-\log\log x) \\ &= \exp\left(\frac{4(\log x)^{\frac{1}{2}}}{\log\log x} + O\left(\frac{(\log x)^{\frac{1}{2}}}{(\log\log x)^2}\right)\right).\end{aligned}$$

Thus, for a suitable choice of $x_0$, it follows that

$$|abc|^2 > P_L(a,b,c)^3\exp\left((12-\delta)\frac{(\log x)^{\frac{1}{2}}}{\log\log x}\right).$$

Moreover, since for a suitable choice of $x_0$,

$$\left(1+\exp\left(\tfrac{1}{2}h_L\right)\right)^2\exp(\theta_L(y)) \leq \exp\left(2(\log x)^{\frac{1}{2}}\right) \leq x^{\delta},$$

it follows that

$$\begin{aligned}
P_L(a,b,c) &\leq \exp(\theta_L(y))|c|^2 \\
&\leq \left(1+\exp\left(\tfrac{1}{2}h_L\right)\right)^2 \exp(\theta_L(y))|b|^2 \\
&\leq \left(1+\exp\left(\tfrac{1}{2}h_L\right)\right)^2 \exp(\theta_L(y))x^{h_L} \\
&\leq x^{h_L+\delta}
\end{aligned}$$

and thus

$$\log P_L(a,b,c) \leq (h_L+\delta)\log x.$$

Next, since we may assume without loss of generality that

$$\log P_0 > \exp(2),$$

and the function

$$\mathbb{R}_{>\exp(2)} \ni z \mapsto \tfrac{z^{\frac{1}{2}}}{\log z} \in \mathbb{R}$$

is strictly monotone increasing, $\frac{12-\delta}{(h_L+\delta)^{\frac{1}{2}}} > 12-\delta'$, and $P_L(a,b,c) > P_0$, it follows that, for a suitable choice of $x_0$,

$$\begin{aligned}
&\exp\left((12-\delta)\frac{(\log x)^{\frac{1}{2}}}{\log\log x}\right) \\
&= \exp\left(\frac{(12-\delta)}{(h_L+\delta)^{\frac{1}{2}}}\cdot\frac{\log\log x+\log(h_L+\delta)}{\log\log x}\cdot\frac{((h_L+\delta)\log x)^{\frac{1}{2}}}{\log((h_L+\delta)\log x)}\right) \\
&> \exp\left((12-\delta')\frac{((h_L+\delta)\log x)^{\frac{1}{2}}}{\log((h_L+\delta)\log x)}\right) \\
&\geq \exp\left((12-\delta')\frac{(\log P_L(a,b,c))^{\frac{1}{2}}}{\log\log P_L(a,b,c)}\right).
\end{aligned}$$

This completes the proof that condition (2) of Lemma 3.2 is satisfied. □

Now we prove Theorem A.

**Proof.** (This proof is similar to [5, Proof of Theorem 2.2].)

First, observe that there exists an $M \in \mathbb{R}_{>0}$ that depends only on $\delta'$ and $\gamma$ such that, for $z \in \mathbb{R}_{>M}$,

$$\tfrac{\log z}{12-\delta'} < z^{\gamma}.$$

Now we apply Lemma 3.2, where we take "$P_0$" to be $\max\{P_0, \exp(M)\}$, to obtain a strict *abc*-triple $(a,b,c)$ such that

$$P_L(a,b,c) > P_0,\ \log P_L(a,b,c) > M$$

and

$$|abc|^2 > P_L(a,b,c)^3 \exp\left((12-\delta')\frac{(\log P_L(a,b,c))^{\frac{1}{2}}}{\log\log P_L(a,b,c)}\right).$$

Then it follows that

$$\frac{\log\log P_L(a,b,c)}{12-\delta'} < (\log P_L(a,b,c))^{\gamma}.$$

Thus, we conclude that

$$\begin{aligned}|abc|^2 &> P_L(a,b,c)^3 \exp\left((12-\delta')\frac{(\log P_L(a,b,c))^{\frac{1}{2}}}{\log\log P_L(a,b,c)}\right)\\ &> P_L(a,b,c)^3 \exp\left((\log P_L(a,b,c))^{\frac{1}{2}-\gamma}\right)\end{aligned}$$

□

# 4 Near Miss *abc*-Triples via Powers of Units

In this section, we prove Theorem B. Note that it follows from Dirichlet's Unit Theorem that $\mathcal{O}_L^{\times} \setminus \mu_L \neq \emptyset$ if and only if $L$ is neither the field of rational numbers nor an imaginary quadratic field. Since we are given $u_0 \in \mathcal{O}_L^{\times} \setminus \mu_L$, it thus follows that $L$ is neither the field of rational numbers nor an imaginary quadratic field.

Now we prove Theorem B.

**Proof.** Let $I$ be a sufficiently large integer ($\geq 2$) such that $\dfrac{I-1}{I+1} > 1-\delta$. Write $\alpha = \prod_{\sigma : L \hookrightarrow \mathbb{C}} (1+|\sigma(u_0)|)$ ($\geq 1$), where $\sigma$ ranges over the embeddings of $L$ into $\mathbb{C}$. Thus, $\alpha$ depends only on $u_0$. Let $q$ be the smallest prime number ($\geq 2$) such that $q > P_0$, $N_L(q) = q^{[L:\mathbb{Q}]} \geq \log \alpha$, and $q\mathcal{O}_L$ is a maximal ideal of $\mathcal{O}_L$. (Note that the existence of such a $q$ follows from Chebotarev's Density Theorem.) If we write

$$l(I) = \#((\mathcal{O}_L/\mathfrak{q}^I)^{\times}) = q^{[L:\mathbb{Q}]\cdot(I-1)}(q^{[L:\mathbb{Q}]}-1),$$

then $0 < l(I) \leq \mathrm{N}_L(q)^I = q^{[L:\mathbb{Q}]\cdot I}$ and $1-u_0^{l(I)} \in \mathfrak{q}^I$. Moreover, it holds that

$$\begin{aligned}\mathrm{N}_L(1-u_0^{l(I)}) &= \prod_{\sigma : L \hookrightarrow \mathbb{C}} |\sigma(1-u_0^{l(I)})|\\ &\leq \prod_{\sigma : L \hookrightarrow \mathbb{C}} \left(1+|\sigma(u_0)|^{l(I)}\right)\\ &\leq \left(\prod_{\sigma : L \hookrightarrow \mathbb{C}} (1+|\sigma(u_0)|)\right)^{l(I)}\\ &\leq \alpha^{\mathrm{N}_L(q)^I}.\end{aligned}$$

Write

$$a := -1,\ b := u_0^{l(I)},\ c := 1 - u_0^{l(I)}.$$

Then since $a, b, c$ are relatively prime in $\mathcal{O}_L$, it follows that

$$\begin{aligned} H_L(a,b,c) &= \prod_{v \in \mathbb{V}^{\mathrm{arc}}(L)} \max\{||a||_v, ||b||_v, ||c||_v\} \\ &\geq \prod_{v \in \mathbb{V}^{\mathrm{arc}}(L)} ||c||_v \\ &= \mathrm{N}_L(c) \end{aligned}$$

(cf. Definition 1.3 (3)) and

$$\begin{aligned} P_L(a,b,c) &= \prod_{\substack{v \in \mathbb{V}^{\mathrm{non}}(L) \\ \mathfrak{p}_v | c}} \mathrm{N}_L(\mathfrak{p}_v) \\ &\leq \frac{\mathrm{N}_L(c)}{\mathrm{N}_L(q)^{I-1}} \\ &\leq \mathrm{N}_L(c) \end{aligned}$$

(cf. Definition 1.3 (2)). Thus, it follows that

$$1 \leq \log \mathrm{N}_L(c) \leq \mathrm{N}_L(q)^I \log \alpha \leq \mathrm{N}_L(q)^{\frac{I+1}{I-1}(I-1)},$$

and hence that

$$(\log \mathrm{N}_L(c))^{1-\delta} \leq (\log \mathrm{N}_L(c))^{\frac{I-1}{I+1}} \leq \mathrm{N}_L(q)^{I-1}.$$

Therefore,

$$\begin{aligned} P_L(a,b,c) &\leq \frac{\mathrm{N}_L(c)}{\mathrm{N}_L(q)^{I-1}} \\ &\leq \frac{\mathrm{N}_L(c)}{(\log \mathrm{N}_L(c))^{1-\delta}} \end{aligned}$$

and thus, since $P_L(a,b,c) \geq q^{[L:\mathbb{Q}]} \geq 2^2 \geq \exp(1)$,

$$\begin{aligned} \mathrm{N}_L(c) &\geq P_L(a,b,c)(\log \mathrm{N}_L(c))^{1-\delta} \\ &\geq P_L(a,b,c)(\log P_L(a,b,c))^{1-\delta}. \end{aligned}$$

Hence, we conclude that

$$H_L(a,b,c) \geq \mathrm{N}_L(c) \geq P_L(a,b,c)(\log P_L(a,b,c))^{1-\delta}.$$

Since $P_L(a,b,c) \geq q > P_0$, this completes the proof of Theorem B. □

Finally, we prove Corollary C.

**Proof.** In Theorem B, we take $u_0 \in \mathcal{O}_L^\times$ so that $u_0$ is not contained in $\mathcal{O}_{L'}^\times$ for any subfield $L'$ of $L$ such that the $\mathrm{rk}_{L'} < \mathrm{rk}_L$. (Here, we note that by elementary Galois theory, there exist only finitely many subfields of $L$.) Then the $u$ obtained by applying Theorem B satisfies the conditions in the statement of Corollary C. □

## 5 Unit-nondegenerate Number Fields

**Lemma 5.1.** *Let $L, L'$ be number fields such that $L' \subsetneq L$. Then $\mathrm{rk}_{L'} = \mathrm{rk}_L$ if and only if $L'$ is totally real, and $L$ is a totally imaginary extension of $L'$ of degree 2.*

**Proof.** Let $r$ (resp. $r'$, $s$, $s'$) be the number of real places of $L$ (resp. real places of $L'$, complex places of $L$, complex places of $L'$). By Dirichlet's unit theorem, $\mathrm{rk}_{L'} = \mathrm{rk}_L$ if and only if $(\#\mathbb{V}^{\mathrm{arc}}(L') =)\ r' + s' = r + s\ (= \#\mathbb{V}^{\mathrm{arc}}(L))$. Let $\pi : \mathbb{V}^{\mathrm{arc}}(L) \to \mathbb{V}^{\mathrm{arc}}(L')$ be the map induced by restriction.

Suppose that $r' + s' = r + s$. Then since $\pi$ is surjective, $\pi$ is also injective. Thus, for any $v \in \mathbb{V}(L)$, $[L_v : L'_{\pi(v)}] = [L : L'] > 1$. In particular, every infinite place of $L'$ (resp. $L$) is real (resp. complex), and $[L : L'] = 2$.

Conversely, if $L'$ is totally real, and $L$ is a totally imaginary extension of $L'$ of degree 2, then a similar argument shows that $\pi$ is bijective, and hence that $\mathrm{rk}_{L'} = \mathrm{rk}_L$. □

Now we prove Proposition D.

**Proof.**

(i) Suppose that $L$ is unit-degenerate. Then it follows from Lemma 5.1 that $L$ is totally complex, and that there exists a totally real subfield $L'$ of $L$ such that $[L : L'] = 2$. Let $D$ be the decomposition group in $\mathrm{Gal}(L/\mathbb{Q})$ of some $v \in \mathbb{V}^{\mathrm{arc}}(L)$. Since $\#(D \cap \mathrm{Gal}(L/L')) = 2 = \#\mathrm{Gal}(L/L')$, and $\#D \leq 2$, it follows that $D = \mathrm{Gal}(L/L')$. Therefore, for any $\sigma \in \mathrm{Gal}(L/\mathbb{Q})$, $\sigma D \sigma^{-1}$ is equal to the decomposition group of $v^\sigma$ in $\mathrm{Gal}(L/\mathbb{Q})$, hence, by a similar argument, also equal to $\mathrm{Gal}(L/L')$. Thus, we conclude that $D = \mathrm{Gal}(L/L')$ is a normal subgroup of $\mathrm{Gal}(L/\mathbb{Q})$, which implies that $D$ is contained in the center of $\mathrm{Gal}(L/L')$ since $\#D = 2$.

Conversely, suppose that the decomposition group $D$ in $\mathrm{Gal}(L/\mathbb{Q})$ of some $v \in \mathbb{V}^{\mathrm{arc}}(L)$ is contained in the center of $\mathrm{Gal}(L/\mathbb{Q})$. Let $L' := L^D$ (i.e., the subfield of $D$-invariants of $L$). Then since $L$ is a totally imaginary Galois extension of $\mathbb{Q}$, and $D$ is contained in the center of $\mathrm{Gal}(L/L')$, it follows from the definition of $D$ that: (a) $L'$ is a Galois extension of $\mathbb{Q}$; (b) $[L : L'] = 2$; (c) the restriction of $v$ to $L'$ is real. Moreover, (a) and (c) imply that $L'$ is totally real. Thus, $L$ is unit-degenerate by Lemma 5.1.

(ii) Suppose that $M$ is unit-degenerate. Then by (i), the decomposition groups of the infinite places of $M$ in $\mathrm{Gal}(M/\mathbb{Q})$ are contained in the center of $\mathrm{Gal}(M/\mathbb{Q})$. Since the decomposition group of an infinite place $v$ of $L$ in $\mathrm{Gal}(L/\mathbb{Q})$ is equal to the images of the decomposition groups in $\mathrm{Gal}(M/\mathbb{Q})$ of the infinite places of $M$ that lie above $v$ and thus contained in the center of $\mathrm{Gal}(L/\mathbb{Q})$, $L$ is unit-degenerate by (i), a contradiction. □

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